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A THERMOMECHANICAL CONSTITUTIVE THEORY FOR ELASTIC COMPOSITES WITH DISTRIBUTED DAMAGE—II. APPLICATION TO MATRIX CRACKING IN LAMINATED COMPOSITES

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Abstract—A continuum mechanics approach is utilized herein to develop a model for predicting the thermomechanical constitution of initially elastic composites subjected to both monotonic and cyclic fatigue loading. In this model the damage is characterized by a set of second-order tensor valued internal state variables representing locally averaged measures of specific damage states such as matrix cracks, fiber–matrix debonding, interlaminar cracking, or any other damage state. Locally averaged history dependent constitutive equations are constructed utilizing constraints imposed from thermodynamics with internal state variables. In Part I the thermodynamics with internal state variables was constructed and it was shown that suitable definitions of the locally averaged field variables led to useful thermodynamic constraints on a local scale containing statistically homogeneous damage. Based on this result the Helmholtz free energy was then expanded in a Taylor series in terms of strain, temperature, and the internal state variables to obtain the stress–strain relation for composites with damage. In Part II, the three-dimensional tensor equations from Part I are simplified using symmetry constraints. After introducing engineering notation and expressing the constitutive equations in the standard laminate coordinate system, a specialized constitutive model is developed for the case of matrix cracks only. The potential of the model to predict degradation of effective stiffness components is demonstrated by solving the problem of transverse matrix cracks in the 90° layer of several crossply laminates. To solve the example problems, the undamaged moduli are determined from experimental data. The internal state variable for matrix cracking is then related to the strain energy release rate due to cracking by utilizing linear elastic fracture mechanics. These values are then utilized as input to a modified laminate analysis scheme to predict effective stiffnesses in a variety of crossply laminates. The values of effective (damage degraded) stiffnesses predicted by the constitutive model are in agreement with experimental results. The agreement obtained in these example problems, while limited to transverse matrix cracks only, demonstrates the potential of the constitutive model to predict degraded stiffnesses.

INTRODUCTION

In Part I it was hypothesized that damage can be modeled by a set of second-order tensor valued internal state variables (ISVs) which represent locally averaged measures of cracking on a scale assumed to be small compared to the boundary value problem of interest. Continuum mechanics[1] was then utilized to construct stress–strain relations in which all components of the degraded modulus tensor can be determined for a given damage state. The intent of Part II is to apply this damage model to the analysis of continuous fiber-reinforced laminated composites. The current paper seeks only to predict axial stiffness as a function of a known damage state. It therefore represents an application only of the stress–strain relations. The all-important ISV growth laws are the subject of ongoing research. Furthermore, as a long range research goal, it is hoped that the characterization of the ISVs for damage will lead to the development of a model for structural failure in terms of the internal state within any local volume in a typical structural component.

Considerable experimental research has been performed in the last decade detailing the growth of damage in laminated composites under both monotonic and cyclic loading conditions[2–8]. The significance of this damage lies in the fact that numerous global material properties such as stiffness, damping and residual strength may be substantially altered during the life of the component. It has been found that the first phase of fatigue is typified by development of a characteristic damage state (CDS)[9] which is composed primarily of matrix cracking in off-axis plies. During the second phase of damage development the CDS contributes to fiber–matrix debonding, delamination, and fiber microbuckling. These phenomena in turn contribute to a tertiary damage phase in which edge delamination and fiber fracture lead to ultimate failure of the specimen[6].

Analytical modeling of stiffness loss in laminated composites with damage appears to be only recently studied. The earliest attempts fall under the general heading of ply discount methods, in which various phenomenological models have been developed to discount ply properties in the presence of damage[10–12]. Axial stiffness reduction and stress distribution in the CDS have also been predicted using a one-dimensional shear lag concept[5]. Kachanov's modulus reduction technique[13] has also been applied to fibrous composites[14] and although promising results were obtained, the model was utilized in uniaxial form only.

Similarly, very little research has been performed to develop ISV growth laws modeling the evolution of damage in laminated composites as a function of load history. Although extended forms of Miner's rule[15] have been proposed for life prediction[16, 17], they are based on simplified microphysical models at this time.

A complex interactive experiment and analysis model (called a mechanistic model) has been proposed[18] for prediction of life of damaged composites. The mechanistic model appears to require numerous experimental results for each geometric layup in order to determine which damage mode results in failure.

Perhaps the most significant attempts to model damage in laminated composites are contained in Refs [19–23]. The first two of these use analytical methods to model a medium with oriented cracks and thus fall under the heading of microphysical techniques. The first of these two uses variational principles to obtain effective moduli for linear elastic cracked plies[19]. The second uses the self-consistent scheme to predict stiffness loss in a single ply as a function of surface area of matrix cracks[20]. It has not, to these authors knowledge, been applied to general laminate analysis. Furthermore, to our knowledge no analytic microphysical technique has yet been developed for predicting stiffness loss in laminated composites when damage modes other than matrix cracking occur.

As stated in Part I, the current model is phenomenological in the sense that the local volume element is modeled experimentally. Another phenomenological model has been proposed in the literature for laminated composites[21–23], and this model has significantly influenced the current model development. Nevertheless, there exist significant differences between these two phenomenological models. The most significant difference is that the damage ISV in Talreja's model is a vector, whereas that proposed herein is a second-order tensor. Support for the second-order tensorial nature of the ISV has been supplied in Ref. [24]. Recently, Talreja has modified his ISV description somewhat to include second-order tensors[25]. Furthermore, the vector-valued model appears at this time to be laminate specific. Although both models have been applied to the combined modes of matrix cracking and internal delamination[26, 27], these attempts must be considered embryonic at this time. It is our contention that both models warrant further study, especially in anisotropic media.

The literature review cited above and in Part I indicates that although substantial progress has been made in damage modeling, the principal results to date deal only with isotropic homogeneous media. It is the contention of these authors that the material heterogeneity and layered orthotropy encountered in laminated composites requires that a more advanced model be developed for these media. The tensorial nature of the damage ISVs proposed in Part I may provide this capability.

In this paper the general constitutive model developed in Part I is specialized for the single damage mode of matrix cracking in the 90° plies of crossply laminates. Properties of a single lamina with known damage are utilized to specify the value of the ISV as a function of damage state. This expression for the matrix crack ISV is then used to predict the damage-degraded axial stiffness of crossply laminates with a variety of stacking sequences. The validity of the constitutive model formulation is verified by comparing the predicted values of stiffness to experimentally measured values for other stacking sequences, thus demonstrating that at least for this case the model is independent of stacking sequence.

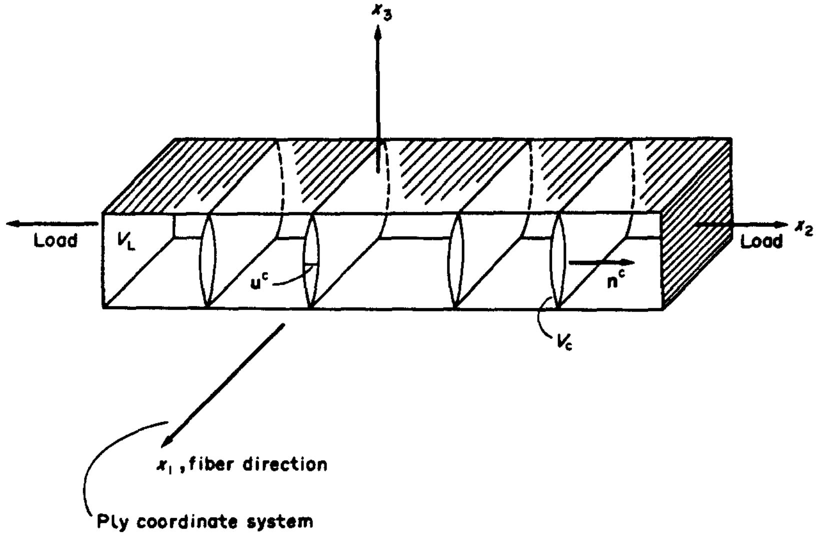

Fig. 1. Transverse matrix cracking in a single ply.

SIMPLIFICATION OF THE MODEL

We now consider the stress–strain relation described in eqns (42)–(45) of Part I (see Appendix A). For the examples to be considered herein, it is assumed that all residual stress components are zero ($\sigma^{R}_{Lij} = 0$), and that there are no tempeature changes ($\Delta T_L = 0$).

Reduction to single-index notation

By incorporating the symmetry of the stress and strain tensors, the quadratic dependence of the Helmholtz free energy on strain, and the Voigt single index notation[28], the constitutive equations reduce to (see Appendix A)

$$\sigma_i = C_{ij}\varepsilon_j + I^{\eta}_{ik}\alpha^{\eta}_{k}. \tag{1}$$

Although we have dropped the subscript L, all quantities in eqn (1) represent locally averaged measures. The subscripts i and j range from 1 to 6, the subscript k ranges from 1 to 9, and the superscript η ranges from 1 to N, the number of damage modes.

At this point, further reductions can be made to the number of unknown constants in eqn (1) only by specifying the material symmetry and specific damage modes of interest.

Material symmetry constraints

Material symmetries may now be utilized to further simplify the constitutive equations. The material in question is assumed to be initially transversely isotropic in the undamaged state on the local scale, where the plane of isotropy is the x_2–x_3 plane shown in Fig. 1. In the undamaged state the modulus tensor C_{ij} is given by[29]

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \tag{2}$$

where $C_{44} = 2(C_{22} - C_{23})$.

It is assumed that the crack induces orthotropy in three planes: the plane of the crack, the plane in which the crack opening displacement $\mathbf{u}^c$ and crack normal $\mathbf{n}^c$ lie, and a third plane which is orthogonal to the first two. Therefore, the damage tensor, I^{1}_{ik} is an orthotropic

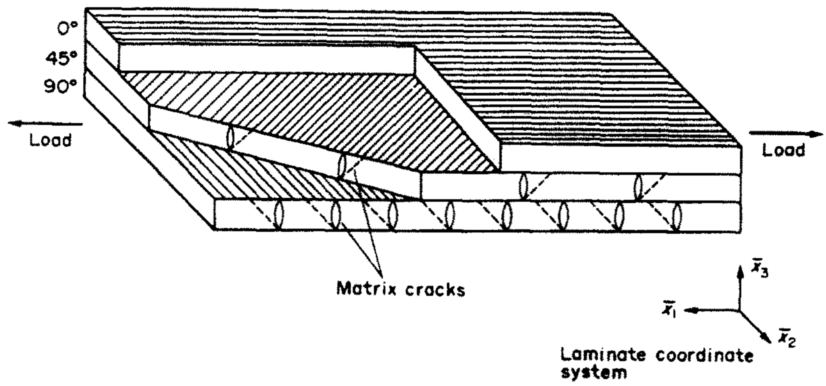

Fig. 2. Matrix cracking in a laminated continuous fiber composite.

tensor containing 15 unknown constants in the coordinates described by the crack geometry (see Appendix B), given by

$$[I^1] = \begin{bmatrix} I^1_{11} & I^1_{12} & I^1_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ I^1_{21} & I^1_{22} & I^1_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ I^1_{31} & I^1_{32} & I^1_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^1_{44} & I^1_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^1_{56} & I^1_{57} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^1_{68} & I^1_{69} \end{bmatrix}. \tag{3}$$

Thus, the complete constitutive equation, eqn (1) (assuming the damage growth law is known) requires the determination of 5 independent material constants for the undamaged modulus tensor $C_{L_{ij}}$, and 15 independent constants for the damage tensor, I^1_{ik}. It should be noted, however, that the planes of these symmetries will not coincide unless the crack displacement $\mathbf{u}^c$ is oriented parallel to the x_1-, x_2-, or x_3-axis in ply coordinates.

Application to matrix cracking in continuous fiber laminates

As discussed in the introduction, the capability of the constitutive model will be demonstrated by considering the case of matrix cracking in continuous fiber laminated composites. An example of this damage state is shown schematically in Fig. 2. In order to apply the proposed constitutive model to this system we first examine the response of a single ply subjected to transverse matrix cracking, as previously shown in Fig. 1. Assuming that the crack geometry is symmetric about normals to each of the ply coordinates, the internal state variable associated with matrix cracking is represented in ply coordinates by

$$[\alpha^1_k] = [0 \quad \alpha^1_2 \quad 0 \quad 0 \quad \alpha^1_5 \quad 0 \quad 0 \quad \alpha^1_8 \quad 0] \tag{4}$$

where the single subscript notation is defined by eqns (A7). This implies that the crack normal $\mathbf{n}^c$ in a single ply is parallel to the local x_2 coordinate. Furthermore, the crack-opening displacement, $\mathbf{u}^c$, may contain three components.

Note that a second-order tensor representation of the internal state variable may be insufficient if the crack displacement $\mathbf{u}^c$ or normal $\mathbf{n}^c$ rotates during the load history. In this case a higher order tensor may be required[30]. However, since the crack is matrix dominated and constrained by fibers, time-dependent rotation is assumed to be negligible and the second-order tensorial representation is considered adequate in the current model. Recent experimental evidence[31] indicates that cracks do indeed change planes sometimes in multiply laminates with several adjacent crossplies at the same orientation. However, the crack plane is essentially straight in each ply, the level at which the local volume is constructed for matrix cracks.

For the single damage mode of matrix cracking described in Fig. 2, eqn (1) reduces to

$$\sigma_i = C_{ij}\varepsilon_j + I^1_{i2}\alpha^1_2 + I^1_{i5}\alpha^1_5 + I^1_{i8}\alpha^1_8. \tag{5}$$

For relatively thin laminates it is useful to apply the conditions of generalized plane stress where the out-of-plane shear stresses σ_4 and σ_5 are neglected. Applying these conditions to eqn (5), imposing the symmetry constraints described in eqns (2) and (3), and using matrix notation results in

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 \\ C_{12} & C_{22} & C_{23} & 0 \\ C_{12} & C_{23} & C_{33} & 0 \\ 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_6 \end{Bmatrix} + [I^1]\{\alpha^1\} \tag{6}$$

where

$$[I^1] = \begin{bmatrix} 0 & I^1_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I^1_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I^1_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^1_{68} & 0 \end{bmatrix}. \tag{7}$$

Note that the fifth column of the coefficient matrix in eqn (5) is zero due to the fact that α^1_5 does not contribute to the in-plane stresses in the generalized plane stress reduction. Furthermore, note that I^1_{12}, I^1_{22} and I^1_{32} are the coefficients of the effect of loss of stiffness on the normal stresses σ_1, σ_2 and σ_3, respectively. Finally, note that I^1_{68} is the coefficient determining the influence of stiffness loss on the in-plane shear stress σ_6. It is apparent from the above equations that for generalized plane stress conditions there are ten unknown material constants to be determined for the case of matrix cracking.

Determination of the I matrix

Theoretically it is possible to determine the I matrix directly from experimental data. This may be accomplished by subjecting test coupons to prescribed deformation histories, removing the deformations, and nondestructively evaluating the damage state. The residual stresses will determine the I matrix. However, in graphite/epoxy laminates this procedure breaks down due to the fact that although the crack surfaces may be determined nondestructively using X-rays and edge replicas, the crack opening displacements cannot be accurately determined experimentally. Therefore, an alternative approach is used herein to evaluate the I matrix.

As described in Appendix C, for the case considered in this paper, at least to a first approximation, it can be shown that

$$\begin{aligned} I^1_{12} &= -C_{12} \qquad & I^1_{22} &= -C_{22} \\ I^1_{32} &= -C_{23} \qquad & I^1_{68} &= -C_{66}. \end{aligned} \tag{8}$$

Therefore, the number of unknown material constants is reduced to a total of six for the case considered herein.

Laminate equations

In order to utilize single lamina equations to characterize the response of multilayered laminates, it is necessary to globally average the local play constitutive equations. This is accomplished herein by imposing the Kirchhoff hypothesis for thin plates[31]. However, higher order plate or shell theories could be utilized also. Generalized plane strain conditions are imposed rather than plane strain because this is consistent with the stress state in eqn (6) (a detailed description of the global averaging is given in Appendix D). The resulting equations are as follows:

$$\{N\} = [A]\{\varepsilon^0\} + \{D\} \tag{9}$$

or

$$\{\varepsilon^0\} = [A^{-1}](\{N\} - \{D\}) \tag{10}$$

where

$$A_{ij} \equiv \sum_{k=1}^{n} (\bar{C}_{ij})_k t_k \tag{11}$$

and

$$[D] \equiv \sum_{k=1}^{n} t_k [\bar{I}^1]_k \{\bar{\alpha}^1\}_k \tag{12}$$

$[\bar{I}^1]_k$ and $\{\bar{\alpha}^1\}_k$ are in laminate coordinates as defined in Appendix E, and $\{\varepsilon^0\}$ contains the laminate midplane strains. Furthermore, k specifies the ply and t_k is the ply thickness. For convenience, we have assumed that no moments are produced by the damage (in the absence of curvature), which is assumed to hold for symmetric laminates.

In order to determine the effective stiffness for any damage state, we evaluate the rate of change of $\{N\}$ with respect to the midsurface strains $\{\varepsilon^0\}$ during unloading, that is

$$S'_{im} = \partial N_i/\partial \varepsilon_j^0 = A_{ij} + \sum_{k=1}^{n} \sum_{j=1}^{9} t_k (\bar{I}_{ij})_k (\partial \bar{\alpha}_j/\partial \varepsilon_m^0)_k \tag{13}$$

where S'_{im} is defined to be the effective stiffness. Experimental work on graphite/epoxy laminates has shown that S'_{11} is very nearly a constant during unloading, implying that, at least as a first approximation for crossply laminates

$$(\partial \bar{\alpha}_1/\partial \varepsilon_1^0)_k \cong \text{constant} \qquad k = 1, \ldots, n. \tag{14}$$

THE INTERNAL STATE VARIABLE FOR MATRIX CRACK DAMAGE

Equation (41) in Part I[1] gave the second-order Taylor series expansion of the local energy per unit volume due to cracking, u_L^c, in terms of strain, ε_{Lij}, temperature ΔT, and the internal state variable (ISV), α_{Lij}^η. For demonstration purposes, the predictions of this paper are being confined to symmetric cross-ply laminates loaded in uniaxial tension with matrix cracks extending straight through the 90° plies. For this case, α_2^1 is the only component of the ISV of interest and is defined (in the ply coordinate system) as

$$\alpha_2^1 = \frac{1}{V_L} \int_{S_2} u_2 \, dS \tag{15}$$

where u_2 is the crack-opening displacement, V_L is the local element volume and S_2 is the surface area of matrix cracks. Furthermore, we will consider only the case of load-up in the fixed grip mode where matrix crack extension occurs at constant strain. Therefore, if the higher order terms in the Taylor series expansion are neglected, the local energy due to cracking reduces to

$$u_L^c = (A + I_{22}^1 \varepsilon_2)\alpha_2^1 \tag{16}$$

where A is a constant. Since eqn (16) applies to load-up in the fixed grip mode, α_2^1 can be related to the strain energy release rate, G_m, for matrix cracking by noting that u_L^c is related to G_m as follows:

$$u_L^c(t) = -\frac{1}{V_L}\int^{S_2(t)} G_m \, \mathrm{d}S \tag{17}$$

where it is assumed that the initial matrix crack surface area is zero and $S_2(t)$ is the surface area at time t. If we make the assumption that the energy stored due to residual damage is negligible, then the constant A in eqn (16) is zero, and equating eqns (16) and (17) yields

$$I_{22}^1 \varepsilon_2 \alpha_2^1 = \frac{-1}{V_L}\int^{S_2(t)} G_m \, \mathrm{d}S \tag{18}$$

for stable crack growth. In order to properly account for crack interaction an expression for G_m will be determined experimentally.

The strain energy release rate due to matrix cracking may be defined as

$$G_m = -\frac{\partial U}{\partial S} \tag{19}$$

where S denotes matrix crack surface area and U is the strain energy of the laminate.

As a first approximation we will assume that all the strain energy released during matrix crack formation occurs in the 90° layer containing the cracks. The strain energy in a 90° layer is defined by

$$U = \tfrac{1}{2}E_{22}\varepsilon_2^2 \tag{20}$$

for a uniform applied strain, ε_2, in an elastic material. Assuming the applied strain to be constant for the fixed grip condition, substituting eqn (20) into eqn (19) results in

$$G_m = -\tfrac{1}{2}V_L \varepsilon_2^2 \frac{\partial E_{22}}{\partial S}. \tag{21}$$

This implies that the effective modulus of the 90° layer changes due to matrix crack formation. It is noted that eqn (21) is similar to the expression for strain energy release rate written in terms of test specimen compliance.

The rule of mixtures yields the following expression for the loading direction modulus of a cross-ply laminate:

$$E_x = \frac{pE_{11} + qE_{22}}{p+q} \tag{22}$$

where p is the number of 0° plies and q is the number of 90° plies. Assuming that matrix cracks are confined to the 90° plies

$$\frac{\partial E_x}{\partial S} = \frac{q}{p+q}\frac{\partial E_{22}}{\partial S}. \tag{23}$$

Substituting eqn (23) into eqn (21) gives

$$G_m = -\tfrac{1}{2}V_L\varepsilon_2^2\frac{p+q}{q}\frac{\partial E_x}{\partial S}. \tag{24}$$

If the right-hand side of eqn (24) is determined experimentally from a laminate that has a 90° layer that is one ply thick, the resulting strain energy release rate can be utilized for other layups by observing that the strain energy release rate for a ply in a layer that is n plies thick is given by (see Appendix F)

$$(G_m)_1 = -\tfrac{1}{2}nV_L\varepsilon_2^2\left(\frac{p+q}{q}\right)\left[\frac{\partial E_x}{\partial S}\right]_{1\text{ ply layer}} \tag{25}$$

where V_L is the volume of a single ply. Equation (25) can now be substituted into eqn (18) to obtain the expression for the ISV of a single 90° ply for matrix crack extension during load-up. The resulting equation is

$$\alpha_2^1 = \tfrac{1}{2}\varepsilon_2\frac{n(p+q)}{I_{22}^1 q}\int_0^{S_2(t)}\frac{\partial E_x}{\partial S}\,\mathrm{d}S. \tag{26}$$

The integral term in eqn (26) can be evaluated as follows:

$$\int_0^{S_2(t)}\frac{\partial E_x}{\partial S}\,\mathrm{d}S = \int_{E_{x0}}^{E_{x1}}\mathrm{d}E_x = E_{x1}-E_{x0} \tag{27}$$

where E_{x0} is the undamaged elastic modulus and E_{x1} is the degraded modulus corresponding to damage state $S_2(t_1)$. Substituting eqn (27) into eqn (26) and rearranging, the ISV for load-up is expressed as

$$\alpha_2^1|_{\text{load-up at }S_2(t_1)} = \tfrac{1}{2}\varepsilon_2\frac{n(p+q)}{q}\frac{E_{x0}}{I_{22}^1}\left(\frac{E_{x1}}{E_{x0}}\bigg|_{S_2(t_1)}-1\right). \tag{28}$$

Although it is possible for matrix crack surface area to increase during unloading, in the current development this effect is assumed to be negligible. Therefore, on unloading α_2^1 depends only on the crack-closure displacement, u_2 in eqn (15), and would go to zero on complete crack closure. Assuming that the crack-closure displacement is linear with strain and the matrix crack surface area is constant, eqn (15) can be rewritten as

$$\alpha_2^1|_{\text{unloading}} = c\varepsilon_2 \tag{29}$$

where c is a constant of proportionality.

The constants in eqns (28) and (29) must be determined from experimental data. Considering a tensile test with a load and unloading cycle, at the instant of load reversal the expressions for the ISV for load-up and unloading must be equal. Therefore, setting eqn (28) equal to eqn (29) and rearranging, gives the following relationship:

$$c|_{S_2(t_1)} = \frac{1}{2}\frac{n(p+q)}{q}\frac{E_{x0}}{I_{22}^1}\left(\frac{E_{x1}}{E_{x0}}\bigg|_{S_2(t_1)}-1\right). \tag{30}$$

It should be noted that all matrix cracking information is contained in the term E_{x1}/E_{x0}. Since eqn (30) applies only to a single 90° ply, E_{x1} was determined from the experimental results of the $[0/90/0]_s$ laminate. The following expression was obtained from a least squares curve fit to the experimental values of E_{x1}/E_{x0} vs S_2:

$$E_{x1}/E_{x0} = 0.9969-0.061607\cdot S_2(t_1)+0.046230\cdot S_2(t_1)^2. \tag{31}$$

Table 1. Material properties for Hercules AS4/3502

Lamina properties		
Longitudinal modulus	E_{11}	$21.0\times10^6\pm2.0\%$ psi
Transverse modulus	E_{22}	$1.39\times10^6\pm2.1\%$ psi
Shear modulus	G_{12}	0.694×10^6 psi
Poisson's ratio	ν_{12}	$0.310\pm3.7\%$
Longitudinal strength	F_{tu1}	$326{,}000\pm3.5\%$ psi
Transverse strength	F_{tu2}	$11{,}085\pm9.8\%$ psi
Long. failure strain	ε_{tu1}	$0.0144\pm4.6\%$ in/in
Tran. failure strain	ε_{tu2}	$0.00773\pm6.7\%$ in/in

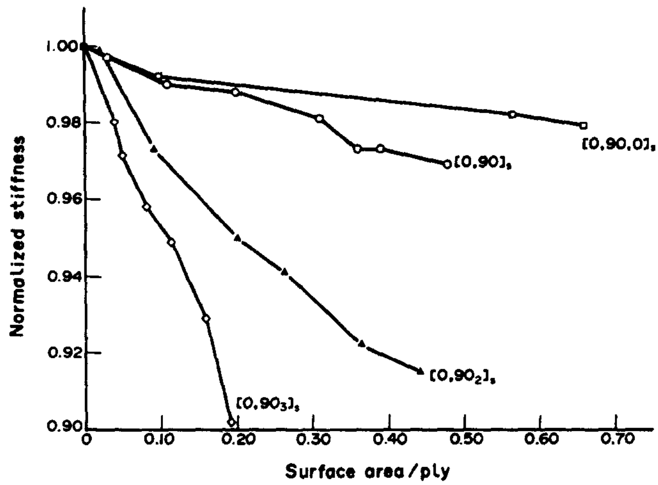

Fig. 3. Experimental results for stiffness loss vs crack area for several crossply laminates.

Recalling eqn (13), it is seen that the effective stiffness of a laminate can be obtained by specifying $\partial\alpha_2^1/\partial\varepsilon_2$. On unloading this is given by eqn (29) to be

$$\partial\alpha_2^1/\partial\varepsilon_2 = c. \tag{32}$$

Therefore, eqns (30) and (31) are used with laminate eqns (13) to predict the effective stiffness of any laminate.

MODEL COMPARISON TO EXPERIMENTAL RESULTS

The model has been utilized to predict the damage-dependent reduced stiffness of several crossply laminates. This has been accomplished by utilizing the laminate stiffness eqns (13), in conjunction with the damage evaluation procedure described in the previous section. The reduced stiffnesses predicted by the model have been compared to experimental results obtained from graphite/epoxy coupons composed of Hercules AS4/3502.

The coupons were obtained from laminates fabricated from prepreg tape using a hot press technique in the Mechanics and Materials Center at Texas A&M University. The laminates were cured according to the procedure recommended by the prepreg tape vendor. Quasi-static tensile tests were conducted on an Instron 1128 screw driven uniaxial testing machine. Matrix crack damage states were evaluated by X-ray radiography and edge replication. Further details of these procedures are contained in Ref. [31]. Initial undamaged lamina properties are given in Table 1. These properties were obtained experimentally from $[0]_8$, $[90]_8$ and $\pm[45]_{2s}$ laminates and are typical for this material system. As discussed in the previous section, the strain energy release rate as a function of surface area was obtained from $[0, 90, 0]_s$ control coupons.

The experimental values of normalized axial stiffness vs matrix crack surface area per 90° ply are shown in Fig. 3. For each laminate, test coupons were quasi-statically loaded

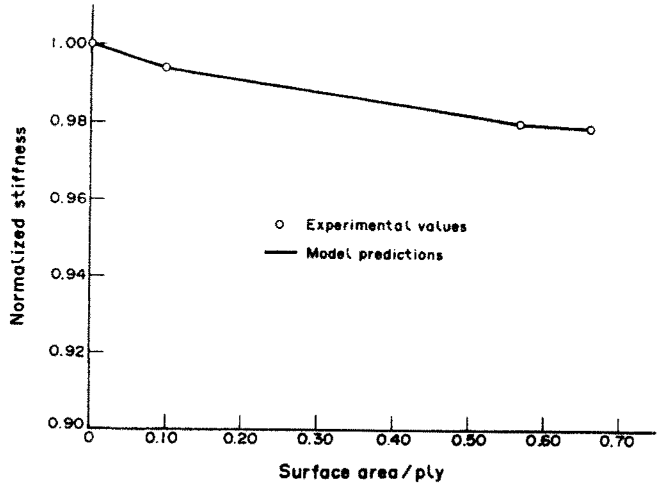

Fig. 4. Model vs experiment for $[0, 90, 0]_s$ laminate.

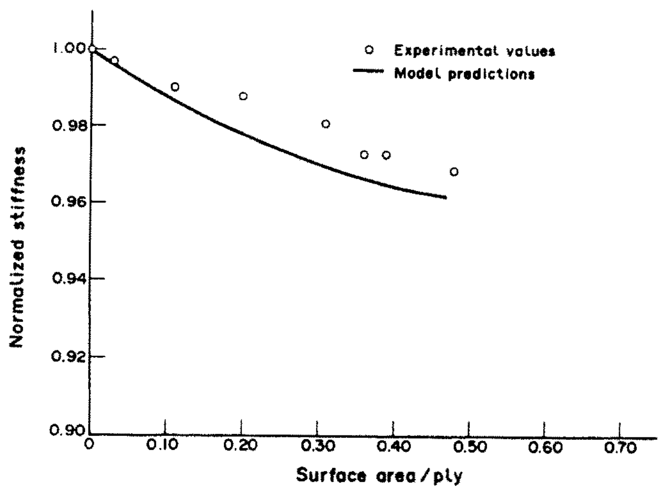

Fig. 5. Model vs experiment for $[0, 90]_s$ laminate.

in an incremental fashion to the matrix crack saturation damage state. At each load step, the matrix crack damage state was documented and the axial modulus was measured by unloading and reloading the coupon. As would be suggested by ply discount theory, the axial stiffness loss increases with increasing number of 90° plies. Also, as would be predicted by shear lag analysis, the number of cracks per inch at the saturation damage state decreases with increasing 90° layer thickness.

Values of effective stiffness for each cross-ply laminate were predicted by the constitutive model using eqns (30), (31), and (13) and the experimentally determined values of matrix crack surface area. Figure 4 presents a comparison of the model predictions to the experimental results for the $[0, 90, 0]_s$ laminate. The excellent agreement between theory and experiment for this laminate was due to the fact that this laminate was used to characterize the strain energy release rate as a function of matrix crack surface area. Figures 5–7 present the comparison between the model predictions and the experimental results for the $[0, 90]_s$, $[0, 90_2]_s$, and $[0, 90_3]_s$ laminates, respectively. As can be seen, the model predictions are in close agreement with the experimental results. The results are quite encouraging given the relatively small stiffness losses of the $[0, 90, 0]_s$ and $[0, 90]_s$ laminates relative to the larger losses experienced by the $[0, 90_2]_s$ and $[0, 90_3]_s$ laminates.

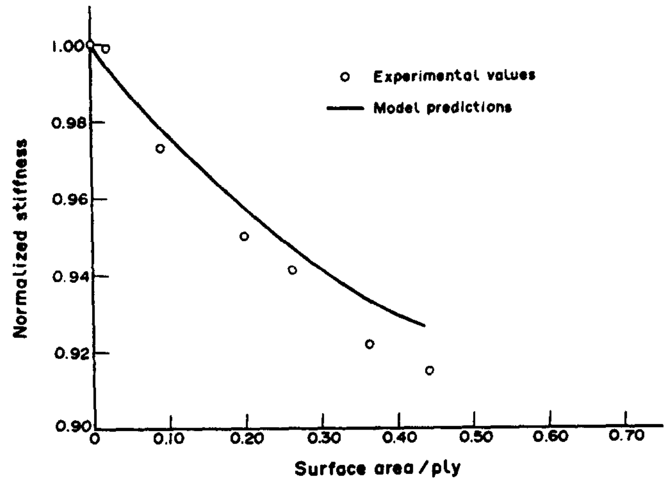

Fig. 6. Model vs experiment for $[0, 90_2]_s$ laminate.

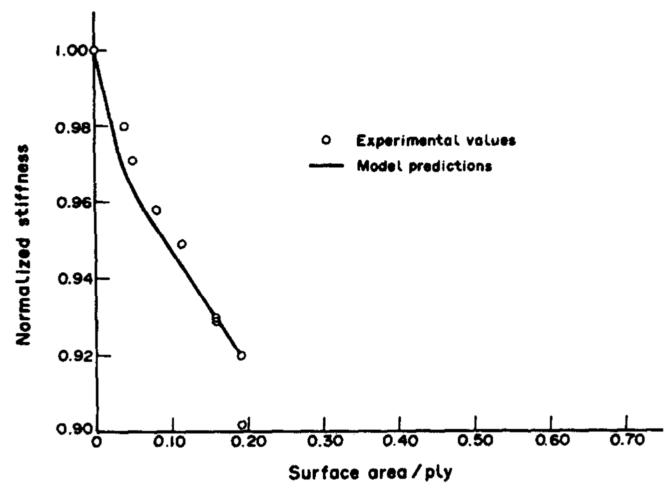

Fig. 7. Model vs experiment for $[0, 90_3]_s$ laminate.

SUMMARY AND CONCLUSIONS

A model for predicting the stiffness loss in laminated composites as a function of microstructural damage has been proposed in this two part paper. In Part I the general theoretical framework was constructed for elastic composites with damage. In Part II the model has been specialized for the case of matrix cracks in crossply laminates. In this process the following key developments have been reported:

(1) material symmetry constraints have been imposed on the damage constant tensor I_{ijkl};

(2) the damage tensor α_{ij} has been reduced for the case of plane stress;

(3) an approximate procedure has been proposed for obtaining the damage constant tensor;

(4) damage dependent laminate equations have been constructed; and

(5) the internal state for any crossply layup has been found to be derivable from energy release rates experimentally obtained from a single layup.

The model has been demonstrated to be accurate in predicting the damage-dependent reduced stiffness of several graphite/epoxy crossply laminates with matrix cracks. While a number of simplifying assumptions were necessary for this model demonstration, most of these assumptions are the same as are typically made by classical lamination theory and do not represent a restriction or limitation to the general applicability of the model. However, the development herein is currently limited to crossply laminates with symmetric damage states. The authors are addressing this limitation by developing damage-dependent laminate equations which account for the curvature produced by non-symmetric damage and the resulting coupling between extension and bending[27]. Finally, the approach described herein depends on the damage state being determined experimentally. This restriction is being addressed by developing damage growth laws which would allow the ISV, and hence the damage state, to be predicted as a function of the loading history of the coupon or structural component.

Current and future development of the model will deal with the following issues:

(1) application of the model to laminates with matrix cracks that are angled or curved rather than extending straight through the 90° layer in the x_1–x_3 plane[24];

(2) application of the model to laminates with both matrix cracks and interply delaminations[27];

(3) application of the model to layups more complex than crossply laminates; and

(4) development of internal state variable growth laws for matrix cracking and interply delamination.

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APPENDIX A: APPLICATION OF SYMMETRY CONSTRAINTS

The damage-dependent constitutive model (eqns (42)–(45) of Part I[1]) is defined as follows:

$$\sigma_{ij} = \sigma_{ij}^{R} + C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{T}) + I_{ijkl}^{\eta}\alpha_{kl}^{\eta} \tag{A1}$$

where σ_{ij} is the local stress tensor, ε_{kl} is the local strain tensor, σ_{ij}^{R} is the residual stress in the absence of strain and temperature change, C_{ijkl} is the undamaged modulus tensor, ε_{kl}^{T} is the thermal strain tensor, α_{kl}^{η} is the internal state variable tensor, and I_{ijkl}^{η} is the damage modulus tensor. Furthermore, we have dropped the subscript L (denoting locally averaged quantities) for convenience.

For demonstration purposes, the residual stress tensor and the temperature change are assumed to be negligible, resulting in

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + I_{ijkl}^{\eta}\alpha_{kl}^{\eta}. \tag{A2}$$

Note that I_{ijkl}^{η} is a fourth-order tensor with 81 coefficients for each value of η. It is assumed here that the constitutive equations given by eqn (A2) are statistically homogeneous. Therefore, the conditions of stress and strain symmetry as well as the existence of an elastic potential can be applied to eqns (A2) to obtain

$$C_{ijkl} = C_{jikl}, \qquad C_{ijkl} = C_{ijlk}, \qquad C_{ijkl} = C_{klij} \tag{A3}$$

and

$$I_{ijkl}^{\eta} = I_{jikl}^{\eta}. \tag{A4}$$

It is most convenient at this point to reindex the constitutive tensors using the Voigt notation[28] where

$$\begin{array}{ll} \sigma_1 \equiv \sigma_{11} & \sigma_4 \equiv \sigma_{23} = \sigma_{32} \\ \sigma_2 \equiv \sigma_{22} & \sigma_5 \equiv \sigma_{13} = \sigma_{31} \\ \sigma_3 \equiv \sigma_{33} & \sigma_6 \equiv \sigma_{12} = \sigma_{21} \end{array} \tag{A5}$$

and

$$\begin{array}{ll} \varepsilon_1 \equiv \varepsilon_{11} & \varepsilon_4 \equiv 2\varepsilon_{23} = 2\varepsilon_{32} \\ \varepsilon_2 \equiv \varepsilon_{22} & \varepsilon_5 \equiv 2\varepsilon_{13} = 2\varepsilon_{31} \\ \varepsilon_3 \equiv \varepsilon_{33} & \varepsilon_6 \equiv 2\varepsilon_{12} = 2\varepsilon_{21}. \end{array} \tag{A6}$$

Furthermore, for all values of η

$$\begin{array}{lll} \alpha_1 \equiv \alpha_{11} & \alpha_4 \equiv \alpha_{23} & \alpha_7 \equiv \alpha_{31} \\ \alpha_2 \equiv \alpha_{22} & \alpha_5 \equiv \alpha_{32} & \alpha_8 \equiv \alpha_{12} \\ \alpha_3 \equiv \alpha_{33} & \alpha_6 \equiv \alpha_{13} & \alpha_9 \equiv \alpha_{21}. \end{array} \tag{A7}$$

Using the contracted notation, eqn (A2) can be written as

$$\sigma_i = C_{ij}\varepsilon_j + I_{ik}^{\eta}\alpha_k^{\eta} \tag{A8}$$

where i and j range from 1 to 6, k ranges from 1 to 9, and η ranges from 1 to N, where N is the number of damage modes.

APPENDIX B: SYMMETRY CONSTRAINTS ON THE DAMAGE MODULUS TENSOR

Consider the following component of internal energy due to cracking:

$$u_1^c \equiv I^1_{ijkl}\varepsilon_{ij}\alpha^1_{kl}. \tag{B1}$$

Since the strain tensor is symmetric

$$I^1_{ijkl} = I^1_{jikl}. \tag{B2}$$

Therefore, there are 54 independent constants in the damage modulus tensor I^1_{ijkl}. Expanding out eqn (B1) thus gives

$$\begin{aligned} u_1^c = {} & I^1_{1111}\varepsilon_{11}\alpha^1_{11}+I^1_{1122}\varepsilon_{11}\alpha^1_{22}+I^1_{1133}\varepsilon_{11}\alpha^1_{33}+I^1_{1123}\varepsilon_{11}\alpha^1_{23} \\ & +I^1_{1132}\varepsilon_{11}\alpha^1_{32}+I^1_{1113}\varepsilon_{11}\alpha^1_{13}+I^1_{1131}\varepsilon_{11}\alpha^1_{31}+I^1_{1112}\varepsilon_{11}\alpha^1_{12} \\ & +I^1_{1121}\varepsilon_{11}\alpha^1_{21}+I^1_{2211}\varepsilon_{22}\alpha^1_{11}+I^1_{2222}\varepsilon_{22}\alpha^1_{22}+I^1_{2233}\varepsilon_{22}\alpha^1_{33} \\ & +I^1_{2223}\varepsilon_{22}\alpha^1_{23}+I^1_{2232}\varepsilon_{22}\alpha^1_{32}+I^1_{2213}\varepsilon_{22}\alpha^1_{13}+I^1_{2231}\varepsilon_{22}\alpha^1_{31} \\ & +I^1_{2212}\varepsilon_{22}\alpha^1_{12}+I^1_{2221}\varepsilon_{22}\alpha^1_{21}+I^1_{3311}\varepsilon_{33}\alpha^1_{11}+I^1_{3322}\varepsilon_{33}\alpha^1_{22} \\ & +I^1_{3333}\varepsilon_{33}\alpha^1_{33}+I^1_{3323}\varepsilon_{33}\alpha^1_{23}+I^1_{3332}\varepsilon_{33}\alpha^1_{32}+I^1_{3313}\varepsilon_{33}\alpha^1_{13} \\ & +I^1_{3331}\varepsilon_{33}\alpha^1_{31}+I^1_{3312}\varepsilon_{33}\alpha^1_{12}+I^1_{3321}\varepsilon_{33}\alpha^1_{21}+I^1_{2311}\varepsilon_{23}\alpha^1_{11} \\ & +I^1_{2322}\varepsilon_{23}\alpha^1_{22}+I^1_{2333}\varepsilon_{23}\alpha^1_{33}+I^1_{2323}\varepsilon_{23}\alpha^1_{23}+I^1_{2332}\varepsilon_{23}\alpha^1_{32} \\ & +I^1_{2313}\varepsilon_{23}\alpha^1_{13}+I^1_{2331}\varepsilon_{23}\alpha^1_{31}+I^1_{2312}\varepsilon_{23}\alpha^1_{12}+I^1_{2321}\varepsilon_{23}\alpha^1_{21} \\ & +I^1_{1311}\varepsilon_{13}\alpha^1_{11}+I^1_{1322}\varepsilon_{13}\alpha^1_{22}+I^1_{1333}\varepsilon_{13}\alpha^1_{33}+I^1_{1323}\varepsilon_{13}\alpha^1_{23} \\ & +I^1_{1332}\varepsilon_{13}\alpha^1_{32}+I^1_{1313}\varepsilon_{13}\alpha^1_{13}+I^1_{1331}\varepsilon_{13}\alpha^1_{31}+I^1_{1312}\varepsilon_{13}\alpha^1_{12} \\ & +I^1_{1321}\varepsilon_{13}\alpha^1_{21}+I^1_{1211}\varepsilon_{12}\alpha^1_{11}+I^1_{1222}\varepsilon_{12}\alpha^1_{22}+I^1_{1233}\varepsilon_{12}\alpha^1_{33} \\ & +I^1_{1223}\varepsilon_{12}\alpha^1_{23}+I^1_{1232}\varepsilon_{12}\alpha^1_{32}+I^1_{1213}\varepsilon_{12}\alpha^1_{13}+I^1_{1231}\varepsilon_{12}\alpha^1_{31} \\ & +I^1_{1212}\varepsilon_{12}\alpha^1_{12}+I^1_{1221}\varepsilon_{12}\alpha^1_{21}. \end{aligned} \tag{B3}$$

We now wish to impose orthotropic symmetry. In order to do this, first rotate 180° about the x_3-axis[28]. The direction cosines for this transformation are

$$[a_{ik'}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{B4}$$

Therefore, since ε_{ij} is a second-order tensor

$$\varepsilon_{k'l'} = \varepsilon_{ij}a_{ik'}a_{jl'} \tag{B5}$$

it follows that

$$[\varepsilon_{k'l'}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & -\varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & \varepsilon_{33} \end{bmatrix}. \tag{B6}$$

Furthermore

$$[\alpha^1_{k'l'}] = \begin{bmatrix} \alpha^1_{11} & \alpha^1_{12} & -\alpha^1_{13} \\ \alpha^1_{21} & \alpha^1_{22} & -\alpha^1_{23} \\ -\alpha^1_{31} & -\alpha^1_{32} & \alpha^1_{33} \end{bmatrix}. \tag{B7}$$

Since u_1^c must be independent of the coordinate system

$$u_1^c = I^1_{p'q'r's'}\varepsilon_{p'q'}\alpha^1_{r's'}. \tag{B8}$$

Substituting eqns (B6) and (B7) into eqn (B8) and comparing this result to eqn (B3) will result in

$$\begin{aligned} & I^1_{1123} = I^1_{1132} = I^1_{1113} = I^1_{1131} = I^1_{2223} = I^1_{2232} = 0 \\ & I^1_{2213} = I^1_{2231} = I^1_{3323} = I^1_{3332} = I^1_{3313} = I^1_{3331} = 0 \\ & I^1_{2311} = I^1_{2322} = I^1_{2333} = I^1_{2312} = I^1_{2321} = I^1_{1311} = 0 \\ & I^1_{1322} = I^1_{1333} = I^1_{1312} = I^1_{1321} = I^1_{1223} = I^1_{1232} = 0 \\ & I^1_{1213} = I^1_{1231} = 0. \end{aligned} \tag{B9}$$

Rotating 180° about the x_2-axis gives

$$[a_{ik'}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{B10}$$

Therefore

$$[\varepsilon_{k'l'}] = \begin{bmatrix} \varepsilon_{11} & -\varepsilon_{12} & \varepsilon_{13} \\ -\varepsilon_{12} & \varepsilon_{22} & -\varepsilon_{23} \\ \varepsilon_{13} & -\varepsilon_{23} & \varepsilon_{33} \end{bmatrix}. \tag{B11}$$

Furthermore

$$[\alpha^1_{k'l'}] = \begin{bmatrix} \alpha^1_{11} & -\alpha^1_{12} & \alpha^1_{13} \\ -\alpha^1_{21} & \alpha^1_{22} & -\alpha^1_{23} \\ \alpha^1_{31} & -\alpha^1_{32} & \alpha^1_{33} \end{bmatrix}. \tag{B12}$$

Substituting eqns (B11) and (B12) into eqn (B8) and comparing to eqn (B3) results in

$$\begin{gathered} I^1_{1112} = I^1_{1121} = I^1_{2212} = I^1_{2221} = I^1_{3312} = I^1_{3321} = 0 \\ I^1_{2313} = I^1_{2331} = I^1_{1323} = I^1_{1332} = I^1_{1211} = I^1_{1222} = I^1_{1233} = 0. \end{gathered} \tag{B13}$$

Rotating 180° about the x_1-axis yields no additional constraints. Therefore, imposition of orthotropic symmetry on I^1_{ijkl} reduces the number of constants to 15. These are

$$\begin{array}{lll} I^1_{11} \equiv I^1_{1111} & I^1_{12} \equiv I^1_{1122} & I^1_{21} \equiv I^1_{2211} \\ I^1_{22} \equiv I^1_{2222} & I^1_{13} \equiv I^1_{1133} & I^1_{31} \equiv I^1_{3311} \\ I^1_{33} \equiv I^1_{3333} & I^1_{23} \equiv I^1_{2233} & I^1_{32} \equiv I^1_{3322} \\ I^1_{44} \equiv I^1_{2323} & I^1_{45} \equiv I^1_{2332} & I^1_{56} \equiv I^1_{1313} \\ I^1_{57} \equiv I^1_{1331} & I^1_{68} \equiv I^1_{1212} & I^1_{69} \equiv I^1_{2121}. \end{array} \tag{B14}$$

Therefore, the orthotropic damage modulus matrix is given by

$$[I^1] = \begin{bmatrix} I^1_{11} & I^1_{12} & I^1_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ I^1_{21} & I^1_{22} & I^1_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ I^1_{31} & I^1_{32} & I^1_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^1_{44} & I^1_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^1_{56} & I^1_{57} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^1_{68} & I^1_{69} \end{bmatrix}. \tag{B15}$$

For the case where

$$[\alpha^1] = [0 \quad \alpha^1_2 \quad 0 \quad 0 \quad \alpha^1_5 \quad 0 \quad 0 \quad \alpha^1_8 \quad 0] \tag{B16}$$

eqn (B15) reduces to

$$[I^1] = \begin{bmatrix} 0 & I^1_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I^1_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I^1_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^1_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^1_{68} & 0 \end{bmatrix}. \tag{B17}$$

APPENDIX C: DETERMINATION OF THE I MATRIX

At a material point in V_L the stress–strain relation in the absence of temperature change is

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{C1}$$

Integrating over the local volume (excluding cracks) gives

$$\sigma'_{Lij} = \frac{1}{V_L}\int_{V_L - V_C} \sigma_{ij}\,dV = \frac{1}{V_L}\int_{V_L - V_C} C_{ijkl}\varepsilon_{kl}\,dV \tag{C2}$$

where σ'_{Lij} is the average stress outside the damage zones. In this section this value of the stress tensor is assumed to be identical to the average stress σ_{Lij}, which includes the average of the stress in the damage zones. Assuming that C_{ijkl} is spatially homogeneous in V_L, the above may be written

$$\sigma'_{Lij} = \frac{C_{ijkl}}{V_L}\int_{V_L - V_C} \varepsilon_{kl}\,dV = \frac{C_{ijkl}}{V_L}\int_{V_L - V_C} \tfrac{1}{2}(u_{k,l}+u_{l,k})\,dV. \tag{C3}$$

Using the divergence theorem on the last term gives

$$\sigma'_{Lij} = C_{ijkl}\left[\frac{1}{V_L}\int_{S_1} \tfrac{1}{2}(u_k n_l + u_l n_k)\,dS + \frac{1}{V_L}\int_{S_2} \tfrac{1}{2}(u_k n_l + u_l n_k)\,dS\right]. \tag{C4}$$

Or, equivalently

$$\sigma'_{Lij} = C_{ijkl}(\varepsilon_{Lkl} - \tfrac{1}{2}\alpha_{kl} - \tfrac{1}{2}\alpha_{lk}). \tag{C5}$$

However, the Taylor series expansion has already given (for isothermal conditions)

$$\sigma'_{Lij} = C_{ijkl}\varepsilon_{Lkl} + I_{ijkl}\alpha_{kl}. \tag{C6}$$

Therefore, equating like terms in eqns (C5) and (C6) gives

$$\begin{aligned} I_{ijkl} &= -C_{ijkl} && k = l \\ I_{ijkl} &= -\tfrac{1}{2}(C_{ijkl}+C_{ijlk}) && k \neq l. \end{aligned} \tag{C7}$$

APPENDIX D: LAMINATE EQUATIONS

The values of generalized plane strain are given by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_z^0 \\ \varepsilon_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} \kappa_x \\ \kappa_y \\ 0 \\ \kappa_{xy} \end{Bmatrix} \tag{D1}$$

where the superscript 0 denotes the midsurface strains and the κ matrix denotes the midsurface curvatures. Under the condition of generalized plane strain there is no warping allowed out-of-plane, which implies that $\kappa_z = 0$.

It is now assumed that no moments or curvatures are imposed and that all laminates studied are symmetric through the thickness (including damage). Therefore, in order to determine the resultant forces, it is necessary only to integrate the given stress state over the laminate thickness to obtain

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \\ N_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \end{Bmatrix} dz \tag{D2}$$

where t is the total thickness of the laminate.

Substituting eqns (A8) and (D1) into eqn (D2) for the case where there are no rotations results in

$$\{N\} = \int_{-t/2}^{t/2} ([\bar{C}]\{\varepsilon^0\} + [\bar{I}^1]\{\bar{\alpha}^1\})\,dz \tag{D3}$$

where $\{N\}$ denotes the force resultants, overbars denote that these quantities are transformed to global coordinates, and $\{\varepsilon^0\}$ represents the mid-surface strains. Note that since transverse cracks are assumed to go completely through the thickness of the cracked plies the stiffness and damage are assumed to be spatially constant through the thickness of a single ply. Therefore, eqn (D3) can be written as

$$\{N\} = \sum_{k=1}^{n} ([\bar{C}]_k(z_k - z_{k-1})\{\varepsilon^0\} + [\bar{I}^1]_k(z_k - z_{k-1})\{\bar{\alpha}^1\}_k) \tag{D4}$$

where k specifies the ply and $z_k - z_{k-1}$ is the thickness of each ply. One can define

$$A_{ij} \equiv \sum_{k=1}^{n} (\bar{C}_{ij})_k(z_k - z_{k-1}) \tag{D5}$$

and

$$\{D\} \equiv \begin{Bmatrix} D_1^1 \\ D_2^1 \\ D_3^1 \\ D_4^1 \end{Bmatrix} = \sum_{k=1}^{n} ([\Gamma^1]_k (z_k - z_{k-1}) \{\bar{\alpha}^1\}_k) \tag{D6}$$

where A_{ij} represents the laminate averaged stiffness matrix and $\{D\}$ is the laminate averaged damage term. Thus, the laminate averaged constitutive equations become

$$\{N\} = [A]\{\varepsilon^0\} + \{D\}. \tag{D7}$$

Experimental testing is often conducted on uniaxial testing machines in which the applied force resultants are input and the strains are experimentally determined output. Therefore, at times, it is more convenient to express the strains in terms of the applied force resultants as follows:

$$\{\varepsilon^0\} = [A]^{-1}(\{N\} - \{D\}). \tag{D8}$$

Note also that moments will be produced even in the absence of strain if the damage state is not symmetric through the thickness. However, for the case considered herein, it will be assumed that all damage states are symmetric, and moments are therefore not considered.

APPENDIX E: TRANSFORMATION EQUATIONS FOR THE DAMAGE TENSOR AND THE DAMAGE MODULUS TENSOR

Consider a coordinate rotation θ in the laminate plane (x_1–x_2 plane) measured clockwise from the ply coordinate system to the laminate coordinate system. For this case the direction cosines are

$$[a_{ik'}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{E1}$$

Recall that since α_{ij}^1 is a second-order tensor

$$\bar{\alpha}_{k'l'}^1 = \alpha_{ij}^1 a_{ik'} a_{jl'}. \tag{E2}$$

Substituting eqn (E1) into eqn (E2) for α_{ij}^1 given by eqn (B16) gives

$$\{\bar{\alpha}^1\} = \begin{Bmatrix} \alpha_{12}^1 \cos\theta \sin\theta + \alpha_{22}^1 \sin^2\theta \\ -\alpha_{12}^1 \sin\theta\cos\theta + \alpha_{22}^1 \cos^2\theta \\ 0 \\ 0 \\ \alpha_{32}^1 \cos\theta \\ 0 \\ \alpha_{32}^1 \sin\theta \\ \alpha_{12}^1 \cos^2\theta + \alpha_{22}^1 \sin\theta\cos\theta \\ -\alpha_{12}^1 \sin^2\theta + \alpha_{22}^1 \sin\theta\cos\theta \end{Bmatrix}. \tag{E3}$$

Furthermore, $\bar{I}_{p'q'r's'}^1$ is given by

$$\bar{I}_{p'q'r's'}^1 = I_{ijkl}^1 a_{ip'} a_{jq'} a_{kr'} a_{ls'}. \tag{E4}$$

Substituting the non-zero components from eqn (B17) into eqn (E4) gives

$$\bar{I}_{p'q'r's'}^1 = I_{12}^1 a_{1p'} a_{1q'} a_{2r'} a_{2s'} + I_{22}^1 a_{2p'} a_{2q'} a_{2r'} a_{2s'} + I_{32}^1 a_{2p'} a_{2q'} a_{3r'} a_{3s'} + I_{68}^1 a_{2p'} a_{1q'} a_{1r'} a_{2s'}. \tag{E5}$$

APPENDIX F: DIMENSIONAL ANALYSIS OF STRAIN ENERGY RELEASE RATES

This appendix develops an approximate dimensional analysis of the strain energy release rate of a cross-ply laminate containing matrix cracks in the 90° plies. The analysis does not attempt to fully address the complexity of the cracking process. For example, the non-linear material effects, crack-tip blunting at the 0/90 interfaces, and the relative thickness of the 0° constraint layers are not included in the simplified analysis. In spite of these limitations, the analysis does adequately account for the influence on the strain energy release rate due to the spacing of pre-existing matrix cracks in the 90° layers and the thickness of the 90° layers. The experimental results

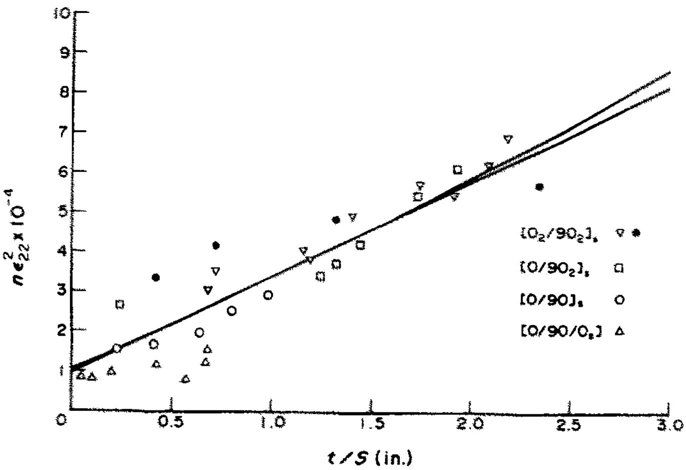

Fig. F1. Experimental strain energy release rate results for a variety of cross-ply laminates (the solid lines are a straight line and quadratic least squares curve fit).

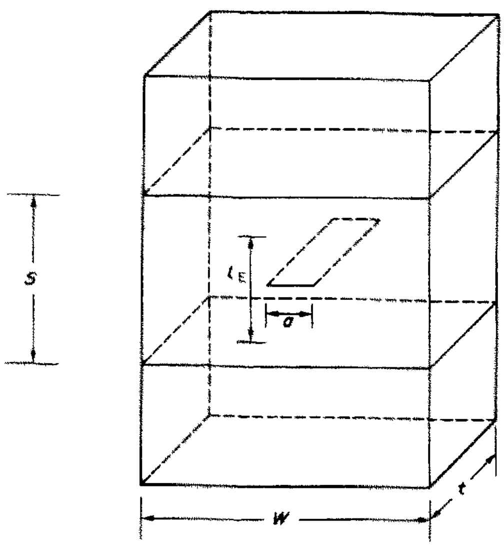

Fig. F2. Crack geometry in the 90° layer.

displayed in non-dimensional form in Fig. F1 suggest that these are the two important dimensional parameters for cross-ply laminates and all other effects are secondary.

Consider the cracked 90° layer shown in Fig. F2. The total strain energy in the region surrounding the crack that includes the strain energy available to be released during crack extension is given by

$$U = U_A + U_B \tag{F1}$$

where U_A is the strain energy ahead of the advancing crack, and U_B is the strain energy behind the advancing crack.

In terms of strain energy density, U_0, eqn (F1) becomes

$$U = (U_0)_A[t(w-a)l_E] + (U_0)_B[tal_E] \tag{F2}$$

where l_E is the effective length of the material from which strain energy will be released by the advancing crack. It should be noted that we are not suggesting that strain energy is only released from a volume of material that is a rectangular parallelepiped. This concept of an effective length is used only as a convenience, as will become evident in the following development. The effective length is a function of both the crack spacing, S, and the thickness of the 90° layer. Therefore, l_E can be expressed as

$$l_E = ct \tag{F3}$$

where c is a non-dimensional function of the crack spacing and the 90° layer thickness. In addition, the strain energy density ahead of the crack and behind the crack can be expressed as functions of the strain energy density in the 90° layer in the absence of cracks multiplied by some dimensionless constant that depends on the existing matrix crack spacing and the thickness of the 90° layer. Written symbolically, then

$$(U_0)_A = U_0 f_A \tag{F4}$$

and

$$(U_0)_B = U_0 f_B \tag{F5}$$

where U_0 is the strain energy density in the 90° layer in the absence of cracks, and f_A and f_B are functions of the crack spacing, S, and thickness, t. Substituting eqns (F3)–(F5) into eqn (F2) yields

$$\begin{aligned} U &= U_0 f_A[ct^2(w-a)] + U_0 f_B[ct^2 a] \\ &= U_0 ct^2[f_A w + a(f_B - f_A)]. \end{aligned} \tag{F6}$$

Strain energy release rate at each crack tip is defined as follows:

$$G = -\frac{1}{2t}\frac{\partial U}{\partial a}. \tag{F7}$$

Substituting eqn (F6) into eqn (F7) thus gives

$$G = \tfrac{1}{2} U_0 ct(f_A - f_B). \tag{F8}$$

Notice that c, f_A and f_B are all functions of the layer thickness and existing matrix crack spacing. Therefore, define a new function f such that

$$f(S, t) = c(f_A - f_B). \tag{F9}$$

Since the function $f(S, t)$ is a dimensionless function, it must be a function of S/t. Therefore

$$c(f_A - f_B) = f(S/t). \tag{F10}$$

Substituting eqn (F10) into eqn (F8) yields the following expression for the available strain energy release rate for matrix cracks:

$$G = \tfrac{1}{2} t[U_0 f(S/t)]. \tag{F11}$$

Therefore, the available strain energy release rate due to matrix cracks in a 90° layer is linearly proportional to the thickness of the 90° layer. The quantity in brackets in eqn (F11) is related to the properties of the composite material system and the laminate stacking sequence. This quantity can be determined from experimental data.

If it is assumed that cracking occurs when the available strain energy release rate is equal to the critical strain energy release rate which is constant for all matrix cracking, then

$$G = G_C. \tag{F12}$$

Furthermore, for a linear elastic material with rigid fibers, the strain energy density is given by

$$U_0 = \tfrac{1}{2} E_{22} \varepsilon_{22}^2. \tag{F13}$$

Substituting eqns (F11) and (F13) into eqn (F12) and solving for strain results in the following expression for the strain in the 90° layer at which matrix crack extension occurs:

$$\varepsilon_{22} = \left[\frac{G_C}{tE_{22} f(S/t)}\right]^{1/2}. \tag{F14}$$

Rearranging gives

$$\frac{4G_C}{tE_{22}\varepsilon_{22}^2} = f(S/t). \tag{F15}$$

Finally, the thickness of the 90° layer is given by

$$t = nt_1 \tag{F16}$$

where t_1 is the thickness of one ply and n is the number of consecutive 90° plies. Substituting eqn (F16) into eqn (F15) results in

$$\frac{4G_C}{t_1 E_{22}}\left[\frac{1}{n\varepsilon_{22}^2}\right] = f(S/t). \tag{F17}$$

If the influence function, $f(S/t)$, is constant for all laminate stacking sequences, then the left-hand side of eqn (F15) will be a function of matrix crack surface area only.

The terms in parentheses on the left- and right-hand sides of eqn (F17) are laminate specific while all other terms are constants. If the function $f(S/t)$ is constant for all laminates then a plot of $n\varepsilon_{22}^2$ vs t/S should be the same for all laminates. The experimental data is plotted in Fig. F1 and as can be seen the data for all laminates follow the same trend curve. Therefore, it can be seen that the available strain energy release rate is a function of the 90° layer thickness and the matrix crack surface area. All other laminate parameters such as the number of consecutive 0° plies results in second-order effects on the energy release rate.